# Iordanskii and Lifshitz-Pitaevskii Forces in the Two-Fluid Model

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*It has been known since the pioneering work of Onsager and Feynman that the statistical mechanics and dynamics of vortices play an essential role in the behavior of superfluids and superconductors. However, the theory of vortices in quantum fluids remains in a most unsatisfactory state, with many conflicting results in the literature. In this paper we review the theory of Thouless, Ao and Niu, which gives an expression for the total transverse force acting on a quantized vortex that is in apparent disagreement with the work of Iordanskii and of Lifshitz and Pitaevskii. In particular, no transverse force proportional to the asymptotic normal fluid velocity was found. We use two-fluid hydrodynamics to study this discrepancy.*

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## 1. A HIERARCHY OF LENGTH SCALES

A comprehensive theory of vortex dynamics in quantum fluids must address phenomena occurring at three different length scales. At the smallest, most microscopic scale, a fully quantum mechanical treatment of a vortex, including its internal structure, interaction with elementary excitations such as phonons and rotons, and interaction with disorder, is required

At the next—what we shall refer to as intermediate—length scale, one would like to regard a vortex as a classical object, subject to a classical equation of motion. Of course, there is no guarantee that we can do this, and, indeed, certain pathologies result from our insistence in doing so, but experience has shown that this classical picture is extremely successful.[1]

There are different ways to formulate the classical approach. If we let $\mathbf{R}(t)$ denote the position in the $xy$ plane of the center of an isolated straight

vortex line, say, as a function of time, we could hypothesize a phenomeno-logical equation of motion of the form

$$M \frac{d^2\mathbf{R}}{dt^2} = -\eta \frac{d\mathbf{R}}{dt} - \gamma \frac{d\mathbf{R}}{dt} \times \mathbf{e}_z + \mathbf{f}_{\mathrm{p}}(\mathbf{R}) + \mathbf{f}_{\mathrm{d}}(\mathbf{R}). \tag{1}$$

Here we have taken the circulation vector $\mathbf{K}$ of the vortex, a vector parallel to the vortex with a magnitude equal to the circulation, to be along the $z$ direction. The first two terms on the right-hand-side of (1) are to include all forces linear in the vortex velocity. The coefficients $M$, $\eta$, and $\gamma$, which describe the vortex effective mass per unit length, viscous damping force per unit length, and nondissipative transverse force per unit length, respectively, are to be determined from a microscopic theory, as are the "pinning" and "driving" forces per unit length $\mathbf{f}_{\mathrm{p}}$ and $\mathbf{f}_{\mathrm{d}}$. The latter may depend on the normal and superfluid densities and velocities, and therefore on the vortex position $\mathbf{R}$, but do not depend on the vortex velocity. In a superfluid, $\mathbf{f}_{\mathrm{p}}$ might describe the force exerted by an externally imposed wire, and $\mathbf{f}_{\mathrm{d}}$ would include the superfluid-velocity-dependent part of the Magnus force (see below) and possibly other vortex-velocity-independent contributions.

At the most macroscopic scale one needs to understand how the forces arising at the intermediate scale act to determine the bulk, experimentally observable properties of quantum fluids. The length scale that usually defines this regime is the characteristic inter-vortex distance, and one is interested in coarse-grained properties of the quantum fluid at scales larger than that distance. In superfluids, this is the regime where the concept of mutual friction applies. Mutual friction refers to a momentum transfer, with both longitudinal and transverse components, between the normal and superfluid parts of a quantum fluid.[1] Although such an interaction is absent in the two-fluid model itself, the presence of vortices mediate a bulk momentum exchange. Similarly, in superconductors one needs to understand how the Lorentz force, pinning forces, and other intermediate-scale forces conspire to determine, say, the Hall effect in the mixed state, which depends on the collective motion of a macroscopic number of vortices.[2]

Our recent work has focused mostly on the intermediate length-scale regime, namely, the determination of the various forces that act on vortices. After a brief general review of that work we shall discuss our new results on the problem of the Iordanskii and Lifshitz-Pitaevskii forces, which have been subjects of considerable controversy.

## 2.   THE TAN THEORY

The Thouless-Ao-Niu (TAN) theory[3] and its generalization to ultra-clean type-II superconductors[4] start with the appropriate exact many-body Hamiltonian and include a pinning potential centered at $\mathbf{R}$, which is also the position of the vortex. The vortex is then dragged with a velocity $\mathbf{V}$, and a transverse force $\mathbf{f}_\perp = \mathbf{e}_z \times \mathbf{V} \oint d\mathbf{l} \cdot \mathbf{j}_c$ is found. Here $\mathbf{j}_c$ is the canonical momentum density, and the line integral is taken around a large circle enclosing the vortex. Evidently, the transverse force is independent of the detailed microscopic structure of the vortex and its interaction with the normal fluid.

In a neutral Bose or Fermi superfluid, $\mathbf{j}_c = \rho_s \mathbf{v}_s + \rho_n \mathbf{v}_n$, and therefore

$$\oint d\mathbf{l} \cdot \mathbf{j}_c = \rho_s K_s + \rho_n K_n. \tag{2}$$

In the normal fluid component, viscosity causes vortex-like circulation to diffuse away to the outer boundary, so $K_n$ vanishes in equilibrium, and

$$\mathbf{f}_\perp = \rho_s \mathbf{K}_s \times \mathbf{V}. \tag{3}$$

In a charged superfluid or superconductor, however, the Hamiltonian contains a current-current interaction term that modifies the velocity operator. The canonical momentum density can be expressed in terms of the physical, gauge-invariant momentum density $\mathbf{j} = \mathbf{j}_c + \frac{e}{c} n \mathbf{A}$, resulting in

$$\oint d\mathbf{l} \cdot \mathbf{j}_c = \oint d\mathbf{l} \cdot (\mathbf{j} - \tfrac{e}{c} n \mathbf{A}) = \rho \, \Phi_0, \qquad \Phi_0 \equiv hc/2e. \tag{4}$$

The second equality in (4) follows from the Meissner effect, which causes $\mathbf{j}$ to vanish at large distances, and from flux quantization. Therefore,

$$\mathbf{f}_\perp = \rho \mathbf{K}_s \times \mathbf{V}, \tag{5}$$

where $\rho = \rho_s + \rho_n$ is the total mass density of the fluid.

## 3.   A HIERARCHY OF CONTROVERSIES

We turn now to a brief discussion of some of the controversies in the theory of intermediate-scale vortex dynamics, focusing on transverse forces, and organized according to the complexity of the quantum fluid in question.

It seems appropriate to start by recalling a result of classical hydrodynamics, the Magnus force: When a vortex with circulation $\mathbf{K}$ is dragged with velocity $\mathbf{V}$ through an ideal fluid of mass density $\rho$, a transverse force per unit length $\mathbf{f}_\perp = \rho \, \mathbf{K} \times \mathbf{V}$ acts on the object doing the dragging. The force

is similar to the lift force on an airplane wing. It is clear from Galilean invariance that if the fluid far from the vortex is not at rest, but has a velocity **v**, then the force is instead

$$\mathbf{f}_\perp = \rho\, \mathbf{K} \times (\mathbf{V} - \mathbf{v}). \tag{6}$$

By analogy, it is reasonable in a neutral Bose superfluid at zero temperature to expect that $\mathbf{f}_\perp = \rho_{\mathrm{s}}\, \mathbf{K}_{\mathrm{s}} \times (\mathbf{V} - \mathbf{v}_{\mathrm{s}})$, where $K_{\mathrm{s}}$ is the quantized circulation. Of course, writing $\rho_{\mathrm{s}}$ here instead of $\rho$ is arbitrary, because these are equal at zero temperature.

The controversy begins in the finite-temperature neutral Bose superfluid, because Galilean invariance allows for a transverse force of the form

$$\mathbf{f}_\perp = a\, \mathbf{K}_{\mathrm{s}} \times (\mathbf{V} - \mathbf{v}_{\mathrm{s}}) + b\, \mathbf{K}_{\mathrm{s}} \times (\mathbf{V} - \mathbf{v}_{\mathrm{n}}), \tag{7}$$

where $a$ and $b$ are parameters. The first term in (7) has a simple classical interpretation: It describes a Magnus-type force originating from the superfluid component of the fluid (remembering that it is the superfluid that is circulating). Thus classical reasoning would suggest that $a = \rho_{\mathrm{s}}$, and Wexler has recently given an elegant proof of this.[5] To our knowledge there have been no criticisms of Wexler's theory.

From this classical point-of-view, again keeping in mind that it is the superfluid that is circulating here, the second term in (7) would be of a non-hydrodynamic origin. If present, it would describe a transverse interaction between the vortex and the excitations—phonons and rotons—of the fluid.

Iordanskii[6] predicted just such an interaction with phonons, and Lifshitz and Pitaevskii[7] (following earlier work by Hall and Vinen[8]) predicted one with rotons. Note, however, that in these works $\mathbf{v}_{\mathrm{s}}$ and $\mathbf{v}_{\mathrm{n}}$ refer to flow velocities *near* the vortex line, whereas our quantities are asymptotic values. But the TAN result (3) implies $a + b = \rho_{\mathrm{s}}$. When combined with Wexler's theory, this implies $b = 0$, i.e., that there are no transverse Iordanskii and Lifshitz-Pitaevskii forces. This apparent discrepancy has motivated us to understand better the interaction between a quantized vortex and the normal fluid in a neutral Bose system, and to calculate the coefficient $b$ in (7) directly. We shall return to this direct calculation below in Section 4.

The next controversy concerns the transverse force in a neutral Fermi superfluid, also described in the TAN theory. According to Kopnin and Kravtsov,[9] low-energy quasiparticles in the vortex core also contribute to the transverse force linear in **V**, a contribution not found by TAN.

Vortices in superconductors inherit all of the above controversies and have additional complexity of their own.[10,11] We will not have space to discuss them further.

## 4.   VORTEX DYNAMICS IN THE TWO-FLUID MODEL

Two ingredients are needed for a microscopic theory of the coefficient $b$ in Eqn. (7). First, one has to solve a vortex-excitation scattering problem. It is probably correct to use the Gross-Pitaevskii equation to do this, even though mean field theory is not expected to hold inside the vortex, because the scattering potential is long-ranged. In fact, the potential is sufficiently long-ranged, and the forward scattering sufficiently singular, that there are several mathematically incorrect scattering calculations present in the literature.[12]

In the scale of lengths greater than the size of the vortex core and less than the mean free path for scattering of excitations by one another, if there is such a range, the dynamics of excitations moving in the slowly varying background of the rotating superfluid is well defined, and in this region the traditional arguments[6,7,13] give a transverse force on the excitations equal to $\rho_n \mathbf{K}_s \times (\mathbf{V} - \mathbf{v}_n)$, where $\mathbf{v}_n$ is the average normal fluid velocity at a distance of the order of the mean free path from the vortex core, as well as a dissipative longitudinal force.

In the region well beyond a mean free path from the vortex core, flow velocities are varying slowly and two-fluid hydrodynamics, which incorporates the basic conservation laws, provides an accurate description. In particular, the two-fluid version of the Navier–Stokes equation gives the force–momentum balance. In this region we have studied how forces generated within a mean free path of the vortex core can be transmitted to large distances.[14] Within a linearized approximation to the two-fluid model the force due to the superfluid flow relative to the vortex is just the ordinary superfluid Magnus force $\rho_s \mathbf{K}_s \times (\mathbf{V} - \mathbf{v}_s)$, where $\mathbf{v}_s$ is the asymptotic value of the superfluid velocity. This is given for superfluids, as it is for ordinary fluids, by the cross terms, between circulation and the superfluid flow, in the momentum flow tensor and the Bernoulli pressure. There seems to be no possible modification of this even for a fermionic superfluid, such as is suggested by the work of Kopnin and Kravtsov,[9] if there is no bulk interaction with a stationary background.

For the normal fluid contributions our considerations are quite similar to those of Hall and Vinen.[8] In the linear regime the force is transmitted by the viscous force (and an accompanying pressure term), which is generated by a flow velocity in the direction of the force that increases as the logarithm of the distance from the vortex core. The asymptotic value of the normal fluid velocity is given by a combination of the value $\mathbf{v}_n$ close to the core, and this logarithmically growing term. The logarithmically growing term has to be cut off at a distance $R_c$ which is determined by the Reynolds

number of the flow, by the spacing between vortices, or by the diffusion length at the frequency of vortex oscillation. In the limit considered by TAN, with vanishingly small normal fluid velocity, and a single vortex in an infinite medium, this cut-off goes to infinity, and $\mathbf{v}_n$ is negligibly small in comparison, so the asymptotic velocity is determined by the logarithmic term, and the force is parallel to the normal fluid flow.

Realistically, the logarithm which the cut-off introduces is not necessarily large compared with other parameters in the problem. Even when the Iordanskii force is not put in explicitly at the inner boundary, there is some transverse force due to normal fluid flow, but this is proportional to $[\ln(R_c/\lambda)]^{-2} v_n$, where $\lambda$ is the mean free path. It is not clear how to match conditions close to the core with conditions in the hydrodynamic region, but if it is assumed that the Iordanskii force causes $\mathbf{v}_n$ close to the core to match the viscous force in the hydrodynamic region, and to be parallel to it, the relative importance of the logarithmic term will depend on the ratio of the kinematic viscosity $\eta/\rho_n$ to the quantum of circulation—a large kinematic viscosity will diminish the importance of the logarithmic term in the flow velocity.

## ACKNOWLEDGMENTS

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